Lecture 4 **2012/10/22**

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1. Motivation

1) Continuous Dependence on Data

- A good mathematical model should have continuous solutions w.r.t. the initial data t_0, x_0 and the system data μ of $f(t, x, \mu)$-small errors in data yield solutions that are close (over some finite time interval) - **Wellposedness!**
- This property is called continuous dependence on data. This continuously dependent property is not possible at points where the solution is not unique! **Why ?**

Remark 4.1. The general form of the IVP is described by

$$\begin{cases} x' = f(t, x, \mu) \\ x(t_0) = x_0 \end{cases}, \qquad (E_\mu)$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_s)^T \in R^s$. Let $z = (x, \mu) \in R^n \times R^s$, then

$$\begin{cases} z' = \tilde{f}(t, z) \\ z(t_0) = z_0 \end{cases} \Leftrightarrow \begin{cases} x' = f(t, x, \mu), & \mu' = 0 \\ x(t_0) = x_0, & \mu(t_0) = \mu \end{cases}. \qquad (H_\mu)$$

It is easy to show (**Homework**):

a) $z(t) = (x(t), \mu(t))$ is a solution of (H_μ) $\Leftrightarrow$ $x(t)$ is a solution of (E_μ) and $\mu(t) \equiv \mu$;

b) If (E_μ) has a unique solution, so does (H_μ).

Then (H_μ) has the same structure to (E). The only difference is their dimensions.

For simplicity of notation, we still consider (E) just regarding t_0 and x_0 as parameter variables.

2) Sensitivity of Variation on Data

- It is natural to ask differentiability for solutions w.r.t. data to characterize the sensitivity of variation on data – **Differentiability Theorem**.

1. Continuous Dependence (Wellposedness)

1) Wellposedness. The IVP (E) is called **wellposed** if there exists a unique solution $x(t, t_0, x_0)$ which depends continuously on (t_0, x_0).

2) Continuous Dependence on Initial Data (t_0, x_0)

The real initial value (t_0, x_0) is obtained by measurement. Suppose the measured initial value is (t_0, x_0) satisfying the following error condition.

$$|t_0 - t_0^0| \leq \frac{h}{2}; \quad \| x_0 - x_0^0 \| \leq \frac{b}{2},$$

where (t_0^0, x_0^0) is the nominal initial value such that the following IVP

$$\begin{cases} x' = f(t, x) \\ x(t_0^0) = x_0^0 \end{cases} \qquad (E_0)$$

has a unique solution $x(t, t_0^0, x_0^0)$ in

$$Q = \{(t, x) \in R \times R^n : |t - t_0^0| \leq a, \ \| x - x_0^0 \| \leq b\}.$$

Let $U = \{(t_0, x_0) \in R \times R^n : |t_0 - t_0^0| \leq \frac{h}{2}, \| x_0 - x_0^0 \| \leq \frac{b}{2}\} \subseteq Q$. Then, we discuss the continuous property of $x(t, t_0, x_0)$ of (E) in the defined domain as follows.

$$G = \{(t, t_0, x_0) \in R \times R \times R^n : |t - t_0^0| \leq \frac{h}{2}; (t_0, x_0) \in U\}.$$

Theorem 4.1 Suppose that $f(t, x)$ is continuous; Lipschitz on Q and $(t_0, x_0) \in U$. Then the solution $x(t, t_0, x_0)$ of (E) is continuous on $(t, t_0, x_0) \in G$.

Proof. First, we construct the Picard approximations $\{x_n(t, t_0, x_0)\}$ ($n \in N^+$) on $t \in [t_0 - h, t_0 + h]$ as follows.

$$x_0(t, t_0, x_0) = x_0, \quad (t, t_0, x_0) \in G$$

$$x_1(t,t_0,x_0)=x_0+\int_{t_0}^{t} f(s,x_0(s,t_0,x_0))\,ds,\quad (t,t_0,x_0)\in G$$

$$\cdots$$

$$x_{n+1}(t,t_0,x_0)=x_0+\int_{t_0}^{t} f(s,x_n(s,t_0,x_0))ds,\quad (t,t_0,x_0)\in G$$

$$\cdots$$

Remark 4.2 For each $n\in N^+$, $x_n(t,t_0,x_0)$ is continuous on (t_0,x_0) for the fixed $t\in[t_0^0-\frac{h}{2},t_0^0+\frac{h}{2}]$. Therefore, $x_n(t,t_0,x_0)$ for each $n\in N^+$ is continuous on $(t,t_0,x_0)\in G$.

Remark 4.3 The reason of defining U and G as above. Since $(t_0,x_0)\in U$ in $\{x_n(t,t_0,x_0)\}$, the interval $|t-t_0|\le h$ varies with t_0. Therefore, the intervals of $\{x_n(t,t_0,x_0)\}$ for each $n\in N^+$ may not be the same in general. For a rigorous sense, we may find

$$[t_0^0-\frac{h}{2},t_0^0+\frac{h}{2}]=\bigcap_{|t_0-t_0^0|\le\frac{h}{2}}[t_0-h,t_0+h]$$

that is a common interval of $\{x_n(t,t_0,x_0)\}$ by using $|t_0-t_0^0|\le\frac{h}{2}$. Therefore, $t\in[t_0^0-\frac{h}{2},t_0^0+\frac{h}{2}]$ is a reasonable common interval. That is, $(t,t_0,x_0)\in G$.

(Cont. of Proof for Theorem 4.1) It is the same to show that the Picard approximations $\{x_n(t,t_0,x_0)\}$ is uniformly convergent to a function $x(t,t_0,x_0)$. That is,

$$\|x_n(t,t_0,x_0)-x(t,t_0,x_0)\|\le\frac{ML^{n-1}}{n!}|t-t_0|^n\le\frac{ML^{n-1}}{n!}(|t-t_0^0|+|t_0^0-t_0|)^n$$

$$\le\frac{ML^{n-1}}{n!}(\frac{h}{2}+\frac{h}{2})^n=\frac{ML^{n-1}}{n!}h^n.$$

Meanwhile, $x(t,t_0,x_0)$ is obviously a solution of E, which is continuous on $(t,t_0,x_0)\in G$ because $x_n(t,t_0,x_0)$ for each $n\in N^+$ is continuous on $(t,t_0,x_0)\in G$. □

2. Differentiability

Theorem 4.2 Suppose that $f(t,x)$ of (E) is of C^2 on Q and $(t_0, x_0) \in U$. Then the solution $x(t, t_0, x_0)$ of (E) is continuously differentiable on $(t, t_0, x_0) \in G$.

Proof. By Theorem 4.1, we take the Picard approximations

$$x_{n+1}(t, t_0, x_0) = x_0 + \int_{t_0}^{t} f(s, x_n(s, t_0, x_0))ds\,, \quad n \in N^+, \quad (t, t_0, x_0) \in G\,,$$

which is continuous on $(t, t_0, x_0) \in G$ for each $n \in N^+$ and uniformly convergent to the solution $x(t, t_0, x_0)$ of (E), which is continuous on $(t, t_0, x_0) \in G$.

Since $f'_x(t,x)$ is also continuously differentiable on Q, we construct an associated Picard matrix sequence as follows.

$$Y_{n+1}(t, t_0, x_0) = I_n + \int_{t_0}^{t} f'_x(s, x_n(s, t_0, x_0))Y_n(t, t_0, x_0)ds\,,$$

where $n \in N^+$ and $(t, t_0, x_0) \in G$. It is similar to show that $\{Y_n(t, t_0, x_0)\}$ is well defined, continuous on $(t, t_0, x_0) \in G$ and uniformly convergent to $Y(t, t_0, x_0)$ that is continuous on $(t, t_0, x_0) \in G$ (**Homework**).

Next, we remark $\dfrac{\partial x_0(t, t_0, x_0)}{\partial x_0} = I_n = Y_0(t, t_0, x_0)$. Then by the definitions of $\{x_n(t, t_0, x_0)\}$ and $\{Y_n(t, t_0, x_0)\}$, we conclude by induction on $n \in N^+$ that

$$\frac{\partial x_n(t, t_0, x_0)}{\partial x_0} = Y_n(t, t_0, x_0)\,, \quad (t, t_0, x_0) \in G\,, \text{ for each } n \in N^+.$$

Therefore, $\{Y_n(t, t_0, x_0)\}$ is a derivative sequence of $\{x_n(t, t_0, x_0)\}$ wrt x_0. Since $\{x_n(t, t_0, x_0)\}$ and $\{Y_n(t, t_0, x_0)\}$ are both uniformly convergent, their limits are

$$\frac{\partial x(t, t_0, x_0)}{\partial x_0} = Y(t, t_0, x_0)\,, \quad (t, t_0, x_0) \in G\,.$$

Then we conclude that $\dfrac{\partial x(t, t_0, x_0)}{\partial x_0}$ is continuous on $(t, t_0, x_0) \in G$.

It is similar to show that

$$\frac{\partial x_{n+1}(t,t_0,x_0)}{\partial t_0} = -f(t_0,x_0) + \int_{t_0}^{t} f'_x(s,x_n(s,t_0,x_0))\frac{\partial x_n(t,t_0,x_0)}{\partial t_0}ds$$

is well defined, continuous and uniformly convergent on $(t,t_0,x_0)\in G$. Then we conclude that $\dfrac{\partial x(t,t_0,x_0)}{\partial t_0}$ is continuous on $(t,t_0,x_0)\in G$ (**Homework**). □

We have simultaneously proved the following theorem.

Theorem 4.3 Suppose that $f(t,x,\mu)$ of (E_μ) is of C^2 on $Q\times D_\mu$, That is, $f'_x(t,x,\mu)$ and $f'_\mu(t,x,\mu)$ are continuously differentiable in $Q\times D_\mu$. Then the solution $x(t,t_0,x_0,\mu)$ of (E_μ) is continuously differentiable on (t,t_0,x_0,μ) in some neighborhood. Moreover, $\dfrac{\partial x(t,t_0,x_0,\mu)}{\partial t_0}$, $\dfrac{\partial x(t,t_0,x_0,\mu)}{\partial x_0}$ and $\dfrac{\partial x(t,t_0,x_0,\mu)}{\partial \mu}$ are respectively the solutions of the following IVP

$$z' = f_x'(t,x(t,t_0,x_0,\mu))z, \quad z(t_0) = -f(t_0,x_0,\mu); \tag{F1}$$

$$z' = f_x'(t,x(t,t_0,x_0,\mu))z, \quad z(t_0) = I_n; \tag{F2}$$

$$z' = f_x'(t,x(t,t_0,x_0,\mu))z + f_\mu'(t,x(t,t_0,x_0,\mu)), \quad z(t_0) = O_{n\times s}. \tag{F3}$$

Proof. By Theorem 4.2, $x(t,t_0,x_0,\mu)$ is continuously differentiable on its arguments (t,t_0,x_0,μ). Then, we take derivatives on both side of

$$x(t,t_0,x_0,\mu) = x_0 + \int_{t_0}^{t} f(s,x(t,t_0,x_0,\mu),\mu)ds$$

to get (F1)-(F3) immediately. □

Remark 4.4 The IVP (F1)-(F3) are all linear systems that are seemly easy to solved. However, they are only conceptual important because $f_x'(t,x(t,t_0,x_0,\mu))$ and $f'_\mu(t,x(t,t_0,x_0,\mu))$ are unknown without knowing solution $x(t,t_0,x_0,\mu)$.

Remark 4.5 The condition f being C^2 both in Theorem 4.2 and Theorem 4.3 can be replaced by a mild condition f being continuously differentiable. However, the present proof no longer workable and a more complicated method will be taken.

3. Continuation Theorem

1) Motivation Example for Continuation

If we apply Picard theorem or Peano theorem to the following Riccati equation:

$$\begin{cases} x' = t^2 + x^2 \\ x(0) = 0 \end{cases},$$

we find:

- $Q_1 = \{(t,x) : |t| \le 1, |x| \le 1\}$, $M = \max_{(t,x)\in Q_1} |t^2 + x^2| = 2 \Rightarrow h_1 = \min\{a, \frac{b}{M}\} = 2$;
- $Q_2 = \{(t,x) : |t| \le 2, |x| \le 2\}$, $M = \max_{(t,x)\in Q_2} |t^2 + x^2| = 8 \Rightarrow h_2 = \min\{a, \frac{b}{M}\} = \frac{1}{4}$.

Some interesting phenomenon arises: $Q_1 \subset Q_2$, but $h_1 > h_2$!

Conclusion:

- This example motivates us that the solution, which is ensured by both Picard theorem and Peano Theorem, can be continuable, e.g. $[-h_2, h_2] \subset [-h_1, h_1]$;
- Both Picard theorem and Peano theorem are local results. It tells nothing about information on the length of existence of interval. We have to develop a new result to characterize continuation properties –**Continuation Theorem**.

2) Some Notions

Definition 4.1 $f : G \to R^n$, where G is an open set of $R \times R^n$, is said to satisfy a **local Lipschitz condition** if for any $(t_0, x_0) \in G$, there exists a neighborhood $(t_0, x_0) \in U \subset G$ such that f satisfies a Lipschitz condition on U.

Definition 4.2 Let $x(t)$ be a solution of the IVP (E) on (α, β). If there exists the other solution $\tilde{x}(t)$ of the IVP (E) on $(\tilde{\alpha}, \tilde{\beta})$ such that

- $(\tilde{\alpha}, \tilde{\beta}) \supset (\alpha, \beta)$, but $(\tilde{\alpha}, \tilde{\beta}) \ne (\alpha, \beta)$;
- $\tilde{x}(t) \equiv x(t)$ for $t \in (\alpha, \beta)$,

we say that $x(t)$ ($t \in (\alpha, \beta)$) is **continuable**, and $\tilde{x}(t)$ is said to be **continuation** of $x(t)$ on $(\tilde{\alpha}, \tilde{\beta})$. We say that a solution $x(t)$ is **non-continuable** if no such continuation exists. That is, (α, β) is a **maximal interval of existence** of $x(t)$. Denoted by $I_{\max} = (\omega_-, \omega_+)$.

2) Continuation Process

Consider the IVP (E), where $f: G \to R^n$ is continuous and local Lipschitz. For the case where $t > t_0$ only, $t < t_0$ is similar.

- $\forall (t_0, x_0) \in G \Rightarrow$ The solution $x(t)$ exists on $I_0 := [t_0, t_0 + h_0]$ with $h_0 > 0$ by Picard theorem, so $x(t_1)$ with $t_1 = t_0 + h_0$ exists and $(t_1, x(t_1)) \in G$;
- If $(t_1, x(t_1)) \in G$ is an interior point of G, then we apply Picard theorem at this point and have a new interval $I_1 := [t_1, t_1 + h_1]$ with $h_1 > 0$, on which $x(t)$ exists. Therefore $x(t_2)$ with $t_2 = t_1 + h_1$ exists and $(t_2, x(t_2)) \in G$;
- If $(t_2, x(t_2))$ is an interior point of G, then we repeat the step 2 to get an interval $I_2 := [t_2, t_2 + h_2]$ with $h_2 > 0, \cdots$; to get $I_j := [t_j, t_j + h_j]$ with $h_j > 0$ on which $x(t)$ exists. Then $x(t)$ is now extended to $\bigcup_{k=1}^{j} I_k$;
- If G is open and bounded, I_j is smaller and smaller because $x(t) \to \partial G$, ∂G is a boundary of G;

  If G is closed and bounded (compact), the continuation will terminate for some step $j = k$ because $(t_k, x(t_k))$ is on ∂G, which can not be applied by Picard theorem anymore.

Remark 4.6 a) We conclude from the process that in all cases, $I_{\max}$ can be found. If G is open, which is usually assumed, then $I_{\max}$ must be open;

b) For f with different $(t_0, x_0) \in G$, $I_{\max}$ might be different! We hope to know what conditions assure the same $I_{\max}$ for all $(t_0, x_0) \in G$. This is a real concern in ODE, which is referred as a **global existence**!!

c) In some case, $x(t)$ will **blow up** at finite time (**finite escape**).

Example $\begin{cases} x' = x^2 \\ x(0) = 1 \end{cases}$ has a solution $x(t) = \dfrac{1}{1-t}$ with $\lim_{t \to 1^-} x(t) = \infty$, $I_{\max} = (-\infty, 1)$.

Remark 4.7 The process of continuation is nothing special except for its asymptotic behavior of solution. This is a real concern of continuation process.

3) Continuation Theorem

Theorem 4.4 (Continuation Theorem) Suppose that G is open in $R \times R^n$, $f : G \to R^n$ is continuous and local Lipschitz. Then every solution of (E) has continuation up to the boundary of G. More precisely, if $x : I_{\max} = (\omega_-, \omega_+) \to R^n$ is the solution passing through $(t_0, x_0) \in G$, then for any compact set $K \subset G$ there exist t_1 and t_2 with $t_1 < t_0 < t_2$ such that $(t_1, x(t_1)) \neq K$, $(t_2, x(t_2)) \neq K$.

Remark 4.8 This theorem says that any solution starting at point in G can be extended continuously to ∂G, which can also be formulized as follows.

$$\lim_{t \to \omega_\pm} \{ d(P(t), \partial G)^{-1} + \| P(t) \| \} = \infty, \qquad \text{(F4)}$$

where $P(t) = (t, x(t))$; d is a distance between $p(t)$ and ∂G; $\| p(t) \| = (t^2 + x^2(t))^{\frac{1}{2}}$.

If $G = R \times R^n$, then ∂G is an empty set. i.e. $d(P(t), \partial G)^{-1} = 0$, (F1) becomes

$$\overline{\lim_{t \to \omega_\pm}} \| P(t) \| = \infty .$$

It means that either $I_{\max} = (-\infty, \infty)$ **(global existence)** or if $I_{\max} = (\omega_-, \omega_+)$, where $\omega_+ < \infty$ and $\omega_- > -\infty$, then $\overline{\lim_{t \to \omega_\pm}} \| x(t) \| = \infty$ **(finite escape)**.

Proof of Theorem 4.4 We prove the case of $[t_0, \omega_+)$ only since $(\omega_-, t_0]$ is similar.

If $\omega_+ = \infty$, then there exists $t_2 > t_0$ s.t. $(t_2, x(t_2)) \in K$ because K is bounded in $R \times R^n$. If $\omega_+ < \infty$. We show it by contradiction. Assume that there exists a compact $K \subset G$ such that $(t, x(t)) \in K$ for all $t \in [t_0, \omega_+)$. Since f is bounded (say M) on the compact set K because f is continuous on K, then we have

$$\| x(t) - x(\tilde{t}) \| \le | \int_{\tilde{t}}^{t} \| f(s, x(s)) \| ds | \le M | t - \tilde{t} |.$$

It shows that f is uniformly continuous on $[t_0, \omega_+)$. Then, $x(\omega_+) = \lim_{t \to \omega_+} x(t)$ exists and is finite. Moreover, $(\omega_+, x(\omega_+)) \in K$ because K is closed. Then, $(\omega_+, x(\omega_+)) \in K \subset G$ is an interior point of G, which shows that it is continuable at ω_+ by Picard theorem. This contradicts the maximality of $I_{\max}$. □

Remark 4.9 If no local Lipschitz, continuation is still workable. However, this continuation is not unique.

Example 4.1 If $x' = f(t, x)$, where $f \in C$ and $\| f(t, x) \| \le M$ for all $(t, x) \in R \times R^n$, show that for any (t_0, x_0), the solution $x(t)$ has $I_{\max} = (-\infty, \infty)$.

Proof For any (t_0, x_0), we have $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds$, and then

$$\| x(t) \| \le \| x_0 \| + | \int_{t_0}^{t} \| f(s, x(s) \| ds | \le \| x_0 \| + M | t - t_0 |.$$

Show by contradiction. If $t_0 \le t < \omega_+$ with $\omega_+ < \infty$, then

$$\| x(t) \| \le \| x_0 \| + M(\omega_+ - t_0) < \infty \quad \Rightarrow \quad \overline{\lim_{t \to \omega_+}} \| x(t) \| < \infty.$$

This contradicts Continuation theorem (see Remark 4.5). It must have $\omega_+ = \infty$. It is similar to show the case of $\omega_- < t \le t_0$ with $\omega_- > -\infty$. □

Example 4.2 All solutions of the Riccati equation $x' = t^2 + x^2$ have a finite escape.

Solution Only show $[t_0, \omega_+)$ with $\omega_+ < \infty$. It is similar to show $\omega_- < t \le t_0$ with $\omega_- > -\infty$. If $\omega_+ \le 0$, then $\omega_+ < \infty$. If $\omega_+ > 0$, then there exists $t_1 > 0$ such that $[t_1, \omega_+) \subseteq [t_0, \omega_+)$. Then we have

$$x'(t) \ge t_1^2 + x^2(t)\,,\ \ t \in [t_1, \omega_+) \ \ \Leftrightarrow \ \ \frac{d\,x(x)}{t_1^2 + x^2(x)} \ge dt\,,\ \ t \in [t_1, \omega_+)\,.$$

Integrating on both sides, we obtain

$$\frac{1}{t_1}[\arctan\frac{x(t)}{t_1} - \arctan\frac{x(t_1)}{t_1}] \ge t - t_1 \ge 0\,,\ \ t \in [t_1, \omega_+)\,.$$

From the above it yields $0 \le t - t_1 \le \frac{\pi}{t_1}$, $t \in [t_1, \omega_+)$. That is, $0 < \omega_+ \le t_1 + \frac{\pi}{t_1} < \infty$. □

7. A Stronger Version of Continuous Dependence on Initial State

1) Lipschitz Condition on a Compact Set

Lemma 4.1 Suppose that G is open in $R \times R^n$, $f: G \to R^n$ is continuous and local Lipschitz. Then for any compact set $K \subset G$ there exists $L > 0$ s.t.

$$\| f(t,y) - f(t,x) \| \le L \| x - y \|, \text{ for all } (t,y),\ (t,x) \in K\,.$$

Proof. By contradiction. If not, there exist $(t_n, x_n),\ (t_n, y_n) \in K$ s.t.

$$\| f(t_n, y_n) - f(t_n, x_n) \| > n \| x_n - y_n \| \, (t_n, x_n)\,,\ \ n \in N^+\,. \tag{F5}$$

Since f is bounded on K with M, it follows that

$$\| x_n - y_n \| \le \frac{2M}{n}\,,\ \ n \in N^+\,. \tag{F6}$$

Since $(t_n, x_n),\ (t_n, y_n) \in K$ have the convergent subsequences by Bolzano-Weierstrass, without loss of generality, say $(t_n, x_n),\ (t_n, y_n) \in K$ themselves. Let $\lim_{n\to\infty}(t_n, x_n) = (\bar{t}, \bar{x}) = \lim_{n\to\infty}(t_n, y_n)$ by (F6). Then, there exists a neighborhood V with $(\bar{t}, \bar{x}) \in V \subseteq K \subset G$ s.t. f satisfies a Lipschitz condition on V by assumption.

However, (F5) contradicts the Lipschitz condition on V . □

Remark 4.10 Lemma 4.1 shows that Local Lipschitz $\Leftrightarrow$ Lipschitz on any compact set.

2) A Stronger version of Continuous Dependence Theorem on Initial Value

Theorem 4.5 Suppose that G is open in $R\times R^n$, $f:G\to R^n$ is continuous and local Lipschitz. Let $x(t,t_0,x_0)$ be a solution of (E) defined on $[t_0,\beta]$, $\beta<\omega_+$, and $x(t,t_0,x_1)$ be a solution of the following IVP:

$$\begin{cases} x'=f(t,x) \\ x(t_0)=x_1 \end{cases}. \tag{E_1}$$

Then for $\forall\,\varepsilon>0$, there exists $\eta>0$ s.t. $\|x_1-x_0\|<\eta \;\Rightarrow\; x(t,t_0,x_1)$ is also defined on $[t_0,\beta]$. Moreover, we have

$$\|x(t,t_0,x_1)-x(t,t_0,x_0)\|\le\varepsilon,\quad t\in[t_0,\beta].$$

Proof. Choose $\varepsilon>0$ small enough such that

$$K=\{(t,x):t\in[t_0,\beta],\ \|x-x(t,t_0,x_0)\|\le\varepsilon\}\subset G.$$

Application of Lemma 4.1 on the compact set K yields that f satisfies

$$\|x(t,t_0,x_1)-x(t,t_0,x_0)\|\le L\|x_1-x_0\|,\text{ for } (t,x)\in K.$$

Taking $\eta=\varepsilon e^{-L(\beta-t_0)}>0$, then we conclude that the interval of existence of $x(t,t_0,x_1)$ in K must not be less than $[t_0,\beta]$. Show it by contradiction. If $x(t,t_0,x_1)$ is defined on $[t_0,\bar\beta]$ with $\bar\beta<\beta$, we have

$$\|x(t,t_0,x_1)-x(t,t_0,x_0)\|\le\|x_1-x_0\|+L\int_{t_0}^{t}\|x(s,t_0,x_1)-x(s,t_0,x_0)\|\,ds$$

$$\Rightarrow\ \|x(t,t_0,x_1)-x(t,t_0,x_0)\|\le\|x_1-x_0\|e^{L(t-t_0)}\quad\text{(Gronwall's inequality)}$$

$$\Rightarrow\ \|x(\bar\beta,t_0,x_1)-x(\bar\beta,t_0,x_0)\|\le\|x_1-x_0\|e^{L(\bar\beta-t_0)}<\eta e^{L(\bar\beta-t_0)}$$

$$=\varepsilon e^{-L(\beta-t_0)}e^{L(\bar\beta-t_0)}=\varepsilon e^{L(\bar\beta-\beta)}<\varepsilon.$$

This shows that the point $(\bar\beta,x(\bar\beta,t_0,x_1))\in K$, which can be extended further by

Picard theorem. This is contradiction. Then applying Gronwall's inequality once more yields

$\| x(t,t_0,x_1) - x(t,t_0,x_0) \| \le \| x_1 - x_0 \| e^{L(t-t_0)} \le \eta e^{L(t-t_0)} < \varepsilon$, whenever $\| x_1 - x_0 \| < \eta$.

Therefore, $x(t,t_0,x_0)$ is a continuous function of x_0. □

Remark 4.11 In fact, $x(t,t_0,x_0)$ is also Lipschitz on x_0 because there exists a constant R s.t.

$$\| x(t,t_0,x_0) - x(t,t_0,x_1) \| \le R \| x_0 - x_1 \|, \quad t \in [t_0, \beta].$$

However, the Lipschitz constant $R = R(t) = e^{L(t-t_0)} \le e^{L(\beta - t_0)}$ depends on the finite interval $[t_0, \beta]$ with $\beta < \infty$. If $\beta = \infty$, this property is not true!!! See $\eta = \varepsilon e^{-L(\beta - t_0)} \to 0$ as $\beta \to \infty$. That is, for any $\varepsilon > 0$, we can't find any $\eta > 0$ to have a desired property!!! Continuous dependence on data for $[t_0, \infty)$ is a global issue that needs an additional condition for sure. It is referred to **Lyapunov stability theory**.

Remark 4.12 The IVP (E) is always wellposed if f is continuous and locally Lipschitz on any finite time interval $[t_0, \beta]$.

6. Summary

- Under mild conditions, the solutions depend continuously on the data for any finite closed interval.
- A good math model should have continuous dependence and differentiability on its data.
- Continuously dependence and differentiability are local results.
- Continuation theorem is a bridge connecting the local and the global.

Homework:

1) If $x' = f(t,x)$, where $f(t,x)$ is continuous and $\| f(t,x) \| \le M \| x \|$ for all $(t,x) \in R \times R^n$, show that for any (t_0, x_0), the solution $x(t)$ has $I_{\max} = (-\infty, \infty)$.

2) All solutions of the Riccati equation $x' = 1 + x^2$ have a blow up at a finite time.